

# Physics of quantum measurement and its interdisciplinary applications

Masahiro Morikawa\*

*Department of Physics, Ochanomizu University, Tokyo 112-0012 JAPAN*

Akika Nakamichi†

*Koyama Observatory, Kyoto-Sangyo University, Kyoto 603-8555 JAPAN*

Quantum dynamics of the collective mode and individual particles on a ring is studied as the simplest model of projective quantum measurement. In this model, the collective mode measures an individual single quantum system. The heart of the model is the wide separation of time scales which yields the distinction of classical and quantum degrees of freedom beyond the standard Gross-Pitaevskii equation. In some restricted cases we derive the Born probability rule. This model is the quantum mechanics version of the effective action method in quantum field theory, which describes the origin of the primordial density fluctuation as classical variables. It turns out that the classical version of this same model successfully describes the dynamics of geomagnetic variation including the polarity flips over 160 million years. The essence of this description is again the coexistence of the wide separated time scales.

---

\* hiro@phys.ocha.ac.jp

† nakamichi@cc.kyoto-su.ac.jp

## I. INTRODUCTION

Quantum mechanics is an excellent theory to describe the Universe and have never failed in laboratory experiments. However in more fundamental level when an individual wave function is considered, especially in relation with delicate measurement processes, the operational description of quantum mechanics is not sufficient and sometime leads us to confusion. A typical situation is the description of the entire Universe by a single wave function. Although the Wheeler-DeWitt equation with any favorite boundary condition formally yields a definite form of the wave function, we have no justified treatment of the wave function. We cannot repeat the evolution process of the Universe nor take any interpretation based on the frequency distribution. More serious problem is the calculation of the primordial density fluctuations in the early Universe. It may be formally possible to calculate the two point correlation function in quantum mechanics for a k-mode inflaton  $\langle \hat{\phi}(x) \hat{\phi}(y) \rangle|_k$  however we have no justification how the spatially inhomogeneity and the statistical power spectrum are related with  $\langle \hat{\phi}(x) \hat{\phi}(y) \rangle|_k$ .

All the above problems seems to stem from the hybrid structure of quantum mechanics: deterministic time evolution described by Schrödinger equation and the stochastic measurement process where the probability enters. The latter process cannot be described only by the former since the latter is not deterministic nor linear. Therefore an extra projection postulate is introduced in quantum mechanics. There have been variety of study on the measurement process so far. We would like to examine a natural physical process based on the collective motion without changing the present formalism of quantum mechanics at all. The measurement process should be physical and it is natural to describe it by the Schrödinger equation at least in the fundamental level. In this paper, we focus on the interaction between the collective mode and the individual degrees of freedom in a simple model.

Collective motion is a common phenomena in various complex systems of many degrees of freedom[1][2]. It means the formation of a localized condensation of the constituent degrees of freedom in the system. This condensation is generally dynamical and appears as a macroscopic degrees of freedom, which we call the order variable. The evolution equation of the order variable is generally different from the one for the individual degrees of freedom and can be non-linear even if the latter is linear. Moreover the appearance of the order variable is often characterized by the separation of its time scale from that of individual degrees of freedom. Actually we successfully analyzed the long-term behavior of geomagnetism based on this line of thought[3]. These two types of degrees of freedom interact with each other and one determines the other self consistently. This self-consistent description brings non-trivial dynamics which was not in the original system. The same is true for quantum theory.

There have been many arguments on the basic part of quantum mechanics[4][5]. We simply focus on the problem how the detector works based on the Schrödinger equation in this paper. An important point we would like to emphasize is that the ordinary decoherence mechanism is not at all sufficient to describe the resultant system state after one-shot measurement. Our goal will be to deduce the result consistent with the standard measurement postulate in quantum mechanics and clarify the physical process actually happening in the detector for quantum measurement. Although a general argument for this purpose utilizing quantum field theory and the generalized effective action has been explored before[6][7][?], present paper is an attempt to study fully within the theory of quantum mechanics. We would like to emphasize how the time scale separation emerges and characterize the detector degrees of freedom in the quantum mechanical argument without resorting to the infinite degrees of freedom of quantum field theory from the beginning.

The next section 2 describes a simple model of measuring apparatus which is described by Schrödinger equation with the self-consistent approximation. We emphasize the distinction between the quantum and classical like behaviors in this system. Section 3 describes the response of the detector by the interaction with the quantum system which is measured, as well as the back reaction of the detector to the system. Section 4 describes the detail of the quantum measurement for the system of general superposition. The last section 5 is devoted to the summary of the work and to the setup of the future studies.

## II. QUANTUM RING MODEL

We consider the wave function for large  $N$  particles moving on a ring of unit radius. No method is known so far to solve this system exactly in general. Furthermore, few exactly-solvable models are exceptional and will not represent general detector apparatus. On the other hand, we have to introduce an environment or its equivalent, after all, to represent the actual detector which decoheres and is non-deterministic. This makes the exactly solvable model unnecessary and allow flexible approximations to solve the model. We use a single tensor product of the wave functions for the individual particles  $\psi_i(t, \theta_i)$ ,  $i = 1, 2, \dots, N$  ( $-\pi \leq \theta_i < \pi$ ), similar to the time-dependent Hartree-Fock approximation. The particles are in the common potential  $V_0(\theta_i) = \cos(\theta_i)^2$  (1) and interact with each other

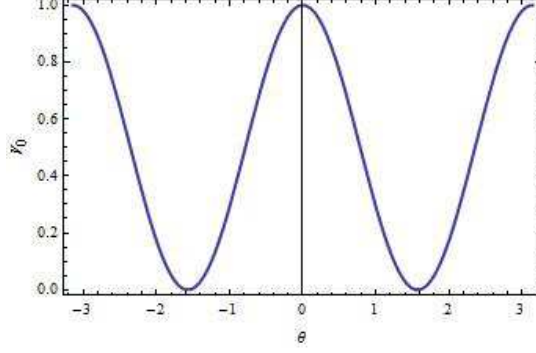


FIG. 1. The graph shows the potential  $V_0(\theta) = \cos(\theta)^2$ . It has two stable points at  $\theta = \pm\pi/2$ . The initial wave functions are set around the unstable point very near to  $\theta = 0$ . The wave function eventually falls down partially into left and partially into right.

with the attractive force[? ].

The potential  $V_0(\theta)$  has two stable points at  $\theta = \pm\pi/2$ . We will set the initial location of many wave functions very near to the unstable point  $\theta = 0$  with small finite variance  $\sigma$ . The attractive interaction can be represented by an extra potential field, which is yielded by all the other particles, as

$$V_{HF}(\theta_i) \equiv \lambda \left( \varphi(t, \theta_i)^2 - \frac{1}{N} |\psi_i(t, \theta_i)|^2 \right) \quad (1)$$

for the  $i$ -th particle ( $\lambda < 0$ ), where

$$\varphi(t, \theta)^2 \equiv \frac{1}{N} \sum_{k=1}^N |\psi_k(t, \theta)|^2 \quad (2)$$

is defined to be the order variable.

Then the equation of motion for the wave function of individual  $i$ -th particle  $\psi_i(t, \theta_i)$  becomes

$$i\hbar \frac{\partial \psi_i(t, \theta_i)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \theta_i^2} \psi(t, \theta_i) + V_0(\theta_i) \psi(t, \theta_i) + V_{HF}(\theta_i) \psi_i(t, \theta_i), \quad (3)$$

where the exchange Fock potential is not necessary in our boson case. The order variable  $\varphi(t, \theta)$  and the individual particle  $\psi_i(t, \theta_i)$  cooperatively determine their evolution. This self-contained structure makes the effective non-linear feature in the set of equations while individual wave function  $\psi_i(t, \theta_i)$  still maintains its linearity since the coupling factor Eq.(1) to  $\psi_i(t, \theta_i)$  in Eq.(3) does not contain itself  $\psi_i(t, \theta_i)$ .

If the attractive force ( $\lambda < 0$ ) is strong enough and dominates the total energy, then the mean field  $\varphi(t, \theta)^2$  tends to be localized in  $\theta$ -space irrespective of the common potential  $V_0(\theta_i)$ . This localized collective mode evolves keeping its locality. Then we naturally call this variable as order variable, which spontaneously violates the spatial ( $\theta$ ) translational invariance.

Since we set the initial location of many wave functions very near to the unstable point  $\theta = 0$  with small finite variance  $\sigma$ , the condition for this localized collective mode to appear may become

$$V_0 < V_{HF}. \quad (4)$$

Furthermore the quantum tunneling probability of the localized order variable  $\varphi(t, \theta)^2$  to penetrate the potential  $V_0(\theta)$  can be extremely small compared with that of a single particle  $\psi_i(t, \theta_i)$ ,

$$\exp \left[ -\frac{\sqrt{Nm\Delta E/2a}}{\hbar} \right] = e^{-(\sqrt{N}-1)\sqrt{m\Delta E/2}(a/\hbar)} \exp \left[ -\frac{\sqrt{m\Delta E/2a}}{\hbar} \right], \quad (5)$$

for large  $N$ , where  $\Delta E$ ,  $a$  respectively represent the typical energy per particle and the typical size of the potential  $V_0$ . This fact yields the separation of time scales in the whole system, as claimed in the introduction. If we choose

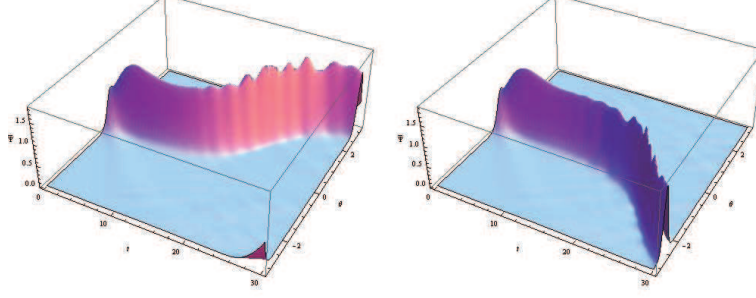


FIG. 2. Time evolution of the order variable  $\varphi(t, \theta)^2$  with  $N = 100$ . We set the initial Gaussian wave functions of the particles at  $\theta \approx 0$  with their peaks have small dispersion  $\sigma$ . The attractive force between the particles makes the whole wave functions to form a coherent cluster which evolves as an almost single Gaussian mode. It eventually falls down the potential valley toward either to the positive (left) or negative (right) side. The destination is fully random reflecting the initial random distribution of the wave functions. However this system can be triggered by the system particle  $i = 0$ , which is set as Eq.(9). The accuracy of the numerical calculations is indicated by the energy conservation within about 0.1%.

the measurement time scale  $T_{measurement}$  as

$$\left[ \exp \left( -\frac{\sqrt{Nm\Delta E/2a}}{\hbar} \right) \right]^{-1} \gg T_{measurement} > \left[ \exp \left( -\frac{\sqrt{m\Delta E/2a}}{\hbar} \right) \right]^{-1}, \quad (6)$$

then the order variable  $\varphi(t, \theta)^2$  behaves as a single classical degrees of freedom while the individual particles evolve as quantum mechanically. Thus the whole system has quantum and classical variables simultaneously and they are consistently described by the set of equations Eqs.(1-3) provided the limited time scale  $T_{measurement}$  is considered.

### III. DETECTOR READOUT - NO TRIGGER

We now demonstrate the time evolution of the wave functions based the Eqs.(1-3). We are especially interested in the correlation between the order variable  $\varphi(t, \theta)^2$  and individual variables  $\psi_i(t, \theta_i)$ ,  $i = 1, 2, \dots, N$ . We prepare the initial wave functions as

$$\psi_i(t = 0, \theta) = \frac{1}{\sqrt{2\pi}s} \exp \left( -\frac{(\theta - \xi_i)^2}{2s^2} \right) e^{i\xi'_i\theta} \quad (7)$$

for  $i = 1, 2, \dots, N$  with  $N = 100$ , and  $\xi_i, \xi'_i$  are random variables which obey Gaussian distributions with the center naught and the dispersion  $\sigma = 10^{-1}$ . Other parameters are  $\hbar = 0.02$ ,  $m = 1$ ,  $s^2 = 10^{-1}$ . Thus all the wave packets are prepared very near around  $\theta \approx 0$ . A typical time evolution of  $\varphi(t, \theta)^2$  is shown in Fig.2.

Individual particles are very unstable around the position  $\theta \approx 0$ , even the peak is exactly located at the top of the potential  $\theta = 0$  since the wave packets Eq.(7) bifurcate toward the both valleys of the potential. However actually in our model, particle are attractive with each other and they tend to be gather to form a localized cluster which evolves coherently. This is the behavior shown in Fig.2.

The localized cluster evolves into either sides of the valley, but not both, for each shot of calculations. This probabilistic feature enters in our model at the initial preparation of the wave functions; the peak locations and the phases of them are chosen randomly. The condition of this collective evolution to occur is given by Eq.(4).

Actually in our case of Eq.(7), this condition becomes

$$\frac{\sigma^2}{2m} + |\sigma V'_0(\sigma)| < \frac{|\lambda|}{2}, \quad (8)$$

which is satisfied in the calculations shown in Fig.2.

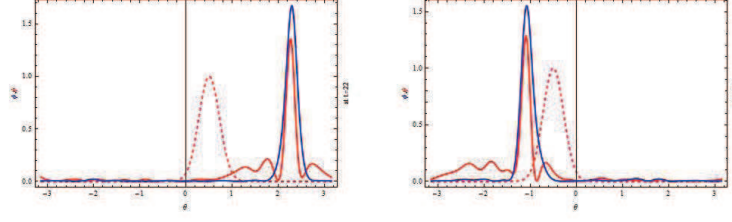


FIG. 3. Snapshots of the system wave function squared  $|\psi_0(t, \theta)|^2$  (solid red line) and the order variable  $|\varphi(t, \theta)|^2$  (solid blue line) at time  $t = 24$ . The initial distribution  $|\psi_0(t, \theta)|^2$  (red broken line) of the system wave function is also shown. The system wave function  $\psi_0(t, \theta)$  initially has Gaussian distribution with its peak at  $\theta = +0.5$  or  $\theta = -0.5$ . The apparatus system is the same as Fig.2. Initially  $\psi_0(t, \theta)$  triggers  $\varphi(t, \theta)^2$  and this  $\varphi(t, \theta)^2$  attracts  $\psi_0(t, \theta)$  to make its peak follow the order variable  $\varphi(t, \theta)^2$ . However, reflecting the fact that the whole system is conservative and the number of particles  $N = 100$  being too small, the peak of the wave function  $\psi_0(t, \theta)$  is not sharp and the whole wave function is dispersed in  $\theta$  space.

#### IV. DYNAMICS OF MEASUREMENT PROCESS

##### A. single trigger

We now apply this model to the dynamics of quantum measurement. We introduce an extra particle,  $i = 0$ , into the above model as the system which is measured and the original particles,  $i = 1, 2, \dots, N$ , are the apparatus. We only measure whether the system particle is located in the positive region or negative region. The evolution equations Eqs.(1-3) hold as before, except that the summation runs on  $i = 0, 1, 2, \dots, N$  in Eq.(2). Original Eq.(2) however, still serves as the definition of the order variable for the apparatus. In summary, the system which is measured is the particle  $i = 0$  and the apparatus is the particles  $i = 1, 2, \dots, N$ . The meter readout is given by Eq.(2). This is a model of non-shot measurement of a quantum state.

We consider in this paper the following one-parameter series of wave functions for the initial state of the system wave function

$$\psi_0(t, \theta_0) = \sin(\alpha) \exp\left(-\frac{(\theta + \frac{1}{2})^2}{2s^2}\right) + \cos(\alpha) \exp\left(-\frac{(\theta - \frac{1}{2})^2}{2s^2}\right). \quad (9)$$

We first consider the case of a single trigger  $\alpha = 0$  and  $\alpha = \pi/2$ . For the trigger  $\alpha = 0$ , for example, the state wave function locates near the right bottom of the potential valley. This equation is solved simultaneously with Eqs.(1-3). The system  $i = 0$  triggers the apparatus particles  $i = 1, 2, \dots, N$ , and attracts them to the right valley. The system wave function itself tends to move with the apparatus particles. Then all the particles fall down toward the right valley without dispersing due to their attractive force. Therefore the system settles down to the positive position with the meter readout Eq.(2) in the positive side. (Fig.3 left)

If the system wave function is set near the left bottom of the potential (*i.e.*  $\alpha = \pi/2$ ), then all the wave functions fall down toward left. In this case, the system settles down to the negative position with the meter readout Eq.(2) in the negative side (Fig.3 right). This dynamical process thus successfully creates the correlation necessary for the quantum measurement process.

The consistency condition between the system wave function and the order variable with respect to the right-or-left positions reduces to the condition that the initial system wave function could actually put sufficient effect for the order variable to roll down toward the direction of the wave function. This condition may become

$$\frac{\sigma}{\sqrt{N}} V' \left( \frac{\sigma}{\sqrt{N}} \right) < \lambda. \quad (10)$$

In the present case in Fig.3, this inequality is satisfied.

The consistency of the locations of system wave function  $\psi_0(t, \theta)$  and the meter reading  $\varphi(t, \theta)^2$  is shown in Fig. 3. The system  $\psi_0(t, \theta)$  is set on the right of the potential eventually settle down toward on the same right side of the potential consistently with the meter readout  $\varphi(t, \theta)^2$ .

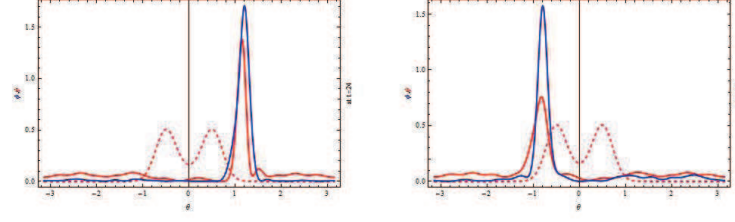


FIG. 4. Same as Fig.3, but the system wave function  $\psi_0(t, \theta)$  initially has evenly superposed two Gaussian distributions with peaks at  $\theta = +1/2$  and  $\theta = -1/2$ .

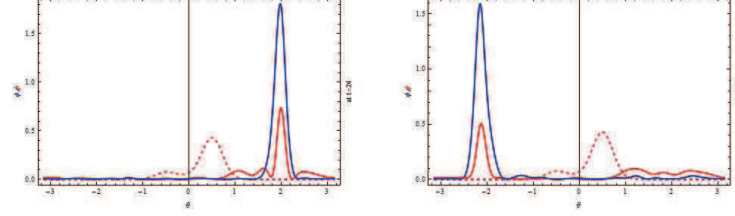


FIG. 5. Same as Fig.3, but the system wave function  $\psi_0(t, \theta)$  initially has unevenly superposed two Gaussian distributions with peaks at  $\theta = +1/2$  and  $\theta = -1/2$ .

### B. even trigger $\alpha = \pi/4$ :

The above faithfulness of the order variable to the final state of the system wave function appears also in the case  $\alpha = \pi/4$ . Fig.4.

The important observation is that the double-peak wave function reduces to a single-peak form, consistent with the order variable. The condition for this reduction to take place is that the time scale of the measurement  $T_{measurement}$  is sufficiently longer than the typical time scale for the system wave function to tunnel the potential. This condition yields the right side of Eq.(6):

$$T_{measurement} > \left[ \exp \left( -\frac{\sqrt{m\Delta E/2a}}{\hbar} \right) \right]^{-1}. \quad (11)$$

### C. non-even trigger

The heart of the quantum measurement is the Born rule. We now study the frequency distribution of the many repetition of measurements for the same initial states. We would like to find a relation between this frequency distribution and the probability read from initial wave function of the system.

We set the initial wave function of the system as the general superposition of the two Gaussian forms,  $0 \leq \alpha \leq \pi/2$ ,

$$\psi_0(t, \theta_0) = \sin(\alpha) \exp \left( \frac{\theta_0 + 0.5}{\Delta\theta} \right) e^{ip_0\theta_0} + \cos(\alpha) \exp \left( \frac{\theta_0 - 0.5}{\Delta\theta} \right) e^{-ip_0\theta_0}. \quad (12)$$

A typical one shot measurement is shown in Fig.5. Then we perform numerical calculations which mimic many repetition of such measurements. A typical result is shown in Fig.6. The graph shows the relative frequency distribution of the many repetition of measurements, with firm correlation between the system state and the meter readout, for the same initial states against the absolute square of the coefficient  $(\sin \alpha)^2$ . The linear dashed line in the figure represents the Born rule of quantum mechanics. The green solid line shows the relative frequency that the apparatus failed to yields the consistent result between the system and the meter readout. The graph shows that our apparatus actually performed quantum measurement with finite accuracy.

We have not very fine-tune the parameters to reproduce the straight line. However generally such tuning or the calibration of the apparatus is necessary for the efficient measurements. Precisely speaking, the balance of the

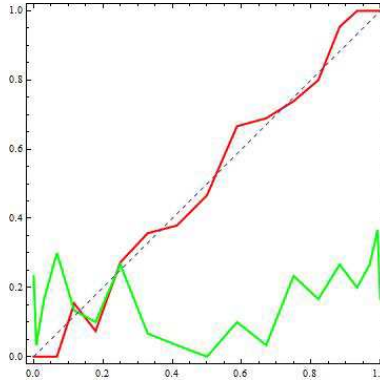


FIG. 6. Red line: The faithfulness of the quantum measurement apparatus. The vertical coordinate is the frequency fraction of the occurrence of the system to be found in the negative region with the meter readout as positive. The horizontal coordinate is  $(\sin \alpha)^2$  of Eq.(12). Green line: The error of the quantum measurement apparatus. The vertical coordinate is the error fraction among the whole run of the measurements.

randomness  $\sigma$  and the strength of trigger  $\lambda$  is essential. If the former dominates the latter, the correlation yields more flat  $N$ -shape, while in the opposite case yields more vertical  $S$ -shape in the graph.

## V. SUMMARY AND DISCUSSIONS

We have discussed the synchronization and clustering of many quantum degrees of freedom and they yield the order variable which has very different time scale from the individual degrees of freedom. Furthermore this order variable is localized dynamical degrees of freedom which does not disperse. Thus it behaves more classically.

We could construct a quantum measuring apparatus utilizing this order variable which couples to the quantum measured system. The wide separation of time scales of them is the essence of this model. Although we could derive very approximate Born rule, further improvements and considerations are necessary. Some of them are as follows.

1. All the calculation was based on the wave function and the Schrödinger evolution, which does not introduce any dissipativity which is considered to be essential for the measurement process. Dissipative and probabilistic nature was only prepared as the initial condition which has random distribution of the wave functions. However this treatment is not complete and we need the description based on the density matrix with dissipation and fluctuations.
2. We utilize time-dependent Hartree-Fock approximation (TDHF) since the actual calculation would be impossible otherwise. However this kind of self-consistent method may neglect any relevant correlations which would be essential for the measurement process. We may need to consider seriously the classical behavior of quantum variables[8].
3. The balance of the randomness  $\sigma$  and the strength of trigger  $\lambda$  was essential in our apparatus. This seems to be a general feature in quantum measurement[6].
4. There is a formalism of quantum measurement which slightly modify the Schrödinger equation including stochasticity and non-linearity at the fundamental level[3]. The basic line of thought to derive the Born rule is the same as ours but in our case we attribute the stochasticity and non-linearity to the detector.

These points should be further studied.

An essential feature of our model is the separation of time scales of the order variable and the system state. Actually the classical version of our model can describe the long-term dynamics of the geomagnetism[3]. In this version the steady global dipole mode with time scale about million years coexists with rapid and local mode with time scale about thousand years. The former corresponds to the apparatus readout and the latter quantum state measured in

our present model.

MM thank Akio Hosoya, Toshiaki Tanaka and Peter Pickl for valuable discussions and important suggestions.

- 
- [1] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge Nonlinear Science Series), Cambridge University Press, Cambridge (2003).
  - [2] G. L. Sewell, *Quantum Theory of Collective Phenomena* (Monographs on the Physics and Chemistry of Materials), Oxford University Press, USA (1990).
  - [3] A. Nakamichi, H. Mouri, D. Schmitt, A. Ferriz-Mas, J. Wicht, and M. Morikawa, Oxford Journals Mathematics & Physical Sciences MNRAS **423**, 2977 (2012)
  - [4] A. Peres, *Quantum Theory: Concepts and Methods* (Fundamental Theories of Physics), Springer (1995).
  - [5] C. J. Isham, *Lectures on Quantum Theory: Mathematical and Structural Foundations*, World Scientific Pub Co Inc (1995).
  - [6] M. Morikawa and A. Nakamichi, Progress of Theoretical Physics **116** 679 (2006).
  - [7] Masahiro Morikawa, preprint [arxiv.org/abs/1211.1739](https://arxiv.org/abs/1211.1739) (2012).
  - [8] P. Pickl, Lett. Math. Phys. **97** 151 (2011).
  - [9] A. Bassi, K. Lochan, S. Satin, et.al., Reviews of Modern Phys. **85** 471 (2013).